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THE ECCENTRIC CIRCLE OF BOSCOVICH.

THE object of the following paper is to draw attention to a general and powerful but quite elementary method of transformation, by means of which the properties of a *conic* may be inferred from those of a *circle*. It has many striking analogies with the method of perspective transformation, and a simple geometrical connection with it. It is, however, of much simpler character for constructive purposes, and though it is intended to show later on how well the method is adapted to serve as an introduction to modern methods, it is hoped that readers will begin by dismissing from their minds all notions of *cross ratios, homographic ranges*, etc., and regard the theorems presented to them from the point of view of a student who has mastered his *Elements of Plane Geometry*.

I. A sketch of Boscovich's own treatment of the conic by means of the eccentric circle.

1. Having defined a conic as a plane curve by means of its *focus* and *directrix*,[1] he proceeds (p. 39) to solve in the most general manner the following problem:—*Having given the focus, directrix, and eccentricity of a conic, to find its intersection with a given straight line* (Figs. 1-5).

Let F be the focus, AB the directrix, and e the eccentricity of the conic; HK the given straight line, cutting AB in H. Take *any* point L on either side of the directrix AB and draw LG perpendicular to AB. Along LG take LS equal to eLG. With centre L and radius LS describe a circle STt.

Join FH. Through L draw a parallel LO to HK, meeting AB in O. Through O draw OZ parallel to FH, cutting the circle STt in Tt. Join LT, Lt Through F draw FP, Fp parallel to LT, Lt, cutting HK in P,p.

Then P,p are the points required.

Draw PD perpendicular to AB.

By similar triangles FPH, LTO—

$$FP : PH :: LT : LO.$$

By similar triangles HPD, OLG—

$$PH : PD :: LO : LG,$$

$$\therefore FP : PD :: LT : LG.$$

But $LT = eLG,$

$$\therefore FP = ePD.$$

$\therefore$ P lies on the conic.

Similarly p lies on the conic.

Conversely, if HK meet the conic in P, and FH, LO, OT be drawn as before, if LT be drawn parallel to FP, it can be proved that

$$LT : LG :: FP : PD.$$

$\therefore LT = eLG.$

$\therefore$ T lies on the circle.

$\therefore$ HK cannot meet the conic in any points but those found by the above construction. We shall follow Dr. Taylor in calling the circle with centre L and radius eLG the *eccentric circle of the point* L.

After remarking in a scholium: "Mirum sane quam fœcunda est hæc constructio, quam Tyroni exercendo apta," he points how the construction may be simplified by taking the point L in certain special positions.

(i.) *On the conic:* in which case the radius of the circle is obviously the focal distance of its centre, and the circle therefore passes through the focus.

(ii.) *At the focus:* in which case the radius is equal to the semi-latus rectum. *This is an important case; it will be treated specially in discussing the connection between Boscovich's transformation and that obtained by conical projection.*

[1] According to Dr. Taylor, Boscovich was the first to write a really complete elementary treatise on conic sections based on the eccentricity or "determining ratio." See *Ancient and Modern Geometry of Conics*, p. lxxii., where the work of Boscovich is commended as "a clear and compact treatise, which for simplicity, depth, and suggestiveness will not readily be surpassed." The references to Boscovich's own treatise (*Elementorum Universæ Matheseos*, tom. iii.) are always to the Venice edition, 1757.

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(iii.) *At the centre of the conic:* in which case the circle becomes the auxiliary circle of the conic.

(iv.) *On the given straight line whose points of intersection with the conic are sought.* This case also is important. The construction obtained in it is that on which *Haslam and Edwards* base their *Conic Sections treated Geometrically* (Longmans, 1881). It will be treated specially in the next section.

2. He then points out that all the points on a conic can be found from the general construction by moving the line HK parallel to itself, the point H traversing the directrix AB.

For if HK keeps its direction, the line LO and therefore the point O remains fixed, while the line OZ turns round the fixed point O.

Taking first the case of the ellipse (Figs. 1, 6), in which, e being less than unity, the eccentric circle lies entirely on the same side of AB as L is. Let H start from a distant point along OA and move to a distant point along OB.

Then OZ will turn from a position as near as we please to OB from one as near as we please to OA. It will at first not cut the circle; then it will touch it at Q; then it will continue to cut it at two points T, t, until it finally touches it again at q, after which it will not meet it.

Hence as the line HK moves parallel to itself it will at first not meet the ellipse; then it will touch it at a point I corresponding to Q; then it will continue to cut the ellipse in points P, p corresponding to T, t, until it finally touches the ellipse again at a point i corresponding to q, after which it will not meet it.

Hence of all straight lines parallel to a given straight line two touch a given ellipse each in one point; of the rest all which lie between these two cut it each in two points, while all those which lie beyond these do not meet it at all.

Proceeding to the parabola (Figs. 2, 7), in which, e being equal to unity, the eccentric circle touches the directrix AB in G. Let H traverse the directrix as before, HK making a constant acute angle AHK with AB. As in the case of the ellipse, OZ, turning about O, it will at first not cut the circle; then it will touch it at a point Q; then it will continue to cut it in two points T, t, however near OZ moves to OA.

Hence as the line HK moves parallel to itself it will at first not meet the parabola; then it will touch it at a point I corresponding to Q; then it will continue to cut it in two points P, p corresponding to T, t, however far H be taken along AB.

Hence of all straight lines parallel to a given straight line which is not perpendicular to the directrix, one touches the parabola in a single point; of the rest each either cuts it in two points or does not meet it at all, according as it lies on the same side of the tangent as the focus is or on the opposite side.

Next let HK be perpendicular to AB. Then O coincides with S and G; the line OZ always cuts the circle in two points T and O (t always coinciding with O) (Fig. 8).

Hence as the straight line HK moves it will always cut the parabola in a single point P corresponding to T, the other point p corresponding to t receding to an infinite distance.

Hence of all straight lines perpendicular to the directrix of a parabola each cuts it in a single point only, the other point receding to an infinite distance, so that it disappears (alterâ intersectione ita in infinitum abeunte ut nusquam jam sit).[1]

In dealing with the hyperbola he considers separately the three cases in which the angle AHK, remaining constant as before, is less than, equal to, or greater than the angle LNn (Figs. 3, 4, 5), which he calls the "angle of equality."

In the first case *two of the parallels touch the curve, one in each branch, the others either cutting one branch in two points or not meeting the curve at all.*

In the second, which is important as leading to the consideration of the asymptotes, *one only of the parallels fails to meet the curve, although it approaches each branch by less than any given distance, however small; of the rest each meets the curve only in a single point, the other point of intersection receding to infinity "ut nusquam jam sit."*

In the third case *all the parallels cut the curve in two points, one on each branch.*

3. The following propositions, which might with advantage have preceded the discussion on the sets of parallels, are then shown to follow from the general construction as simple corollaries.

A straight line cannot cut a conic section in more than two points, or touch it in more than one.

[1] This expression *ut nusquam jam sit* (or *sint*) is of very frequent occurrence in Boscovich.

For OZ cannot cut the *circle* in more than two points, or touch it in more than one.

The line joining the focus to the point where a secant cuts the directrix is equally inclined to the lines joining the focus to the points where it cuts the directrix.

For since OZ is equally inclined to LT, Lt, it follows that FH is equally inclined to FP, Fp. The limiting case is noted.

4. An ingenious proof involving chord properties only is given of the proposition that the locus of the mid points of a set of parallel chords is a straight line.

We supply the demonstration and figure for the case of the ellipse. Slight modifications only are required for the parabola and hyperbola.

With the same construction as in Fig. 1 draw FA perpendicular to HPp, meeting the directrix in I. Bisect Pp, Tt in R,V. Join LV, RI.

By similar triangles

$$\text{HP} : \text{HF} :: \text{OL} : \text{OT}$$
$$\text{HF} : \text{H}p :: \text{O}t : \text{OL}$$
$$\therefore \text{HP} : \text{H}p :: \text{O}t : \text{OT}$$
$$\therefore \text{HP} + \text{H}p : \text{HP} :: \text{O}t + \text{OT} : \text{O}t$$
$$\therefore \text{HR} : \text{HP} :: \text{OV} : \text{O}t.$$

But $\text{HP} : \text{HF} :: \text{OL} : \text{OT}$,

and $\text{HF} : \text{HA} :: \text{OL} : \text{OV}$

$$\therefore \text{HR} : \text{HA} :: \text{OL}^2 : \text{O}t.\text{OT}.$$

But OT.Ot is constant.

$\therefore$ R lies on a straight line through I.

5. The special case of the general construction in which the centre L of the eccentric circle lies on the straight line whose points of intersection with the conic are required, is specially discussed and utilised, amongst other things, for drawing tangents from an external point, and for proving the ratio of the rectangles under the segments of two chords through a point, independent of its position, so long as their directions remain unchanged, but as the next section is devoted to this very case no details of his proofs are here given. E. M. LANGLEY.

THE GREEK GEOMETERS BEFORE EUCLID.

The learned professor of Higher Geometry in the university of Genoa has undertaken to write a treatise on *The Exact Sciences in Ancient Greece*.[1] The treatise will consist of five books, the first being devoted to the period of preparation, the second to the period of greatest splendour, the third to the period of decadence or the epoch of the commentators, the fourth will trace the mathematical discoveries incidentally made by the Greek astronomers and surveyors, and the fifth will treat of the arithmetic of the Greeks.

The first book (*I geometri greci precursori di Euclide*) has been printed in the Memoirs of the Royal Academy of Sciences of Modena, vol. x. Series II, 1893; and it is proposed to give here a brief synopsis of its contents. It is hoped that this abstract may have the effect of directing attention to Professor Loria's treatise, and also to some of the other important works on the history of Mathematics which have appeared within recent years.[1]

After a few general observations on the pre-Euclidean epoch, the author quotes an important passage[2] from the History of Geometry written by Eudemus of Rhodes, one of Aristotle's pupils, and then passes on to

THALES AND THE IONIC SCHOOL.

Thales was born in Miletus about 640 B.C., and lived to more than ninety years of age. His family emigrated from Phoenicia, or more probably from Boeotia, and he himself, when a young man, travelled to Egypt for commercial purposes. He profited by the culture which a country so old and so advanced in civilisation offered to him, and after returning to his native land continued to occupy himself with business and politics. It was only towards the close of his life that he gave himself up to scientific speculations. He acquired great renown throughout the whole Hellenic world by his prediction (to what degree of accuracy is not known) of a solar eclipse, which occurred on the day of a pitched battle between the Medes and the Lydians. Such a prediction presents to the historian two problems:—

1. The determination of the date of this astronomical phenomenon. It must have been the 30th September 610, the 31st July 597, or the 28th May 585 B.C. Most historians choose the last of these dates.

[1] *Le Scienze Esatte nell' Antica Grecia*, di Gino Loria (1893).

[1] The titles of three of these are:—

Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. i. (1880), vol. ii. (1892).

Paul Tannery, *La Géométrie Grecque* (1887).

G. J. Allman, *Greek Geometry from Thales to Euclid* (1889).

[2] Preserved by Proclus (450 A.D.) in his *Commentary on the First Book of Euclid.*

2. What amount of astronomical knowledge does the prediction presuppose on the part of Thales?

It is almost certain that Thales composed no work on pure Mathematics, and it is difficult to know what credit ought to be given to the statements in the extract from Eudemus regarding the discoveries of Thales. These statements are that he was the first to demonstrate that—

(*a*) A circle is bisected by its diameter.

(*b*) The angles at the base of every isosceles triangle are equal.

(*c*) When two straight lines cut one another the vertically opposite angles are equal.

According to Pamphila, a female historian of the time of Nero, Thales was the first to inscribe in a circle a right-angled triangle.

Eudemus also attributes to Thales the first part of the theorem of Euclid I. 26, seeing that he made use of it in the problem of determining the distance of a ship from the shore.

Other writers credit Thales with the solution of the problem: To determine the height of a pyramid (obelisk?) knowing the length of the shadow it casts. This was done either by observing the moment when our shadow is of the same length as ourselves; or by placing a staff at the extremity of the shadow cast by the pyramid and showing that the height of the pyramid was to the length of the staff in the same ratio as their shadows.

Though Thales was not in the habit of gathering round him a group of persons to whom he might expound, or with whom he might discuss the results of his meditations, yet it is customary to consider him as the head of a school. The members of that school are Mandryatus of Priene, Mamercus the brother of the poet Stesichorus, Anaximander, Anaximenes, Anaxagoras.

J. S. MACKAY.

ON APPROXIMATIONS AND REDUCTIONS.

1. It may not be generally known how much more rapidly such operations as multiplying or dividing by π, or reducing metres to yards, etc. can be effected by the methods of "Practice" than by direct multiplication and division.

2. For example, the commonly used approximation $3\frac{1}{7}$ for the value of π, though an excellent first approximation, is only true to three significant figures; but (i) $3\frac{1}{7} - \frac{1}{800}$ is true to five significant figures, and thus yields a second approximation by a correction of the first. The value (ii) $3 + \frac{1}{8} + \frac{1}{600}$, which is slightly higher, may be used as a useful check.

($(3 + \frac{1}{7} - \frac{1}{800} = 3{\cdot}1416\frac{1}{14}$; $3 + \frac{1}{8} + \frac{1}{600} = 3{\cdot}1416\frac{2}{3}$).

EXAMPLE.—*Find the circumference of a circle when the diameter is* 37·158.

(i)	37·158	(ii)	37·158
	111·474		111·474
	5·308		4·645
	116·782		·619
	·046		116·738
	116·736		

A still higher degree of accuracy, which will seldom be required, is given by the approximation—

$$3 + \frac{1}{7} - \frac{1}{800} - \frac{1}{700000} = 3{\cdot}141592\frac{6}{7}.$$

Two other expressions, each equal to 3·1416, are worthy of notice—

(iii) $(3 + \frac{1}{7})(1 - {\cdot}0004)$; (iv) $3(1 + \frac{1}{250})(1 - \frac{1}{4000} - \frac{1}{6000})$.

The advantage of (iii) is that if the product of several factors including π be required, the approximation $3\frac{1}{7}$ can be used first, and, at the end of the multiplication, the correction ($\frac{4}{10000}$ of the product) subtracted. The advantage of (iv) lies in its containing 3 as a factor, which is useful in the reduction of degrees to circular measure.

To illustrate their use we take the same numerical example as before—

(iii)	37·158	(iv)	37·158
			1·8579
	111·474		
	5·308		39·0159
	116·782		117·0477
	·047		·2926
	116·735		·0195
			116·7356*

The multiplier, based on (iv), for reducing degrees to circular measure is $\frac{1}{60}(1 + \frac{1}{250})(1 - \frac{1}{4000} - \frac{1}{60000})$.

3. The reciprocal of π to eight significant figures is ·31830989. To this we have the following approximation.

(i) $\frac{1}{3} - (\frac{1}{100} + \frac{1}{2500}) = {\cdot}3183\frac{1}{3}$.
(ii) $\frac{1}{3} - (\frac{1}{100} + \frac{1}{2500} + \frac{1}{80000}) = {\cdot}31831\frac{1}{3}$.
(iii) $\frac{1}{3} - (\frac{1}{100} + \frac{1}{2500} + \frac{1}{400000}) = {\cdot}318308\frac{1}{3}$.
(iv) $\frac{1}{3}(1 - \frac{1}{22})(1 + {\cdot}0004) = {\cdot}318309\frac{1}{11}$.

EXAMPLE.—*The circumference of a circle is* 116·74. *Find the diameter.*

(i) 116·74	(ii) 116·74	(iii) 116·74	(iv) 116·74
			22) 38·913
38·913	38·913	38·913	6·59
			11·1
1·167	1·167	1·167	
·584	·584	·584	−1·769
	2	3	
37·162*			37·144
	37·160*	37·159*	·015
			37·159

The multiplier for converting circular measure into degrees to seven significant figures is 57·29578, to which we have the approximation—

$$60(1 - \tfrac{1}{20} + \tfrac{1}{200}) = 57{\cdot}3.$$

4. There are cases in which a decimal approximation is the best. Thus to five significant figures—

$$\pi^2 = 10\ (1 - {\cdot}01304); \qquad \frac{1}{\pi^2} = \frac{1}{10}\ (1 + {\cdot}01321).$$

5. Since 1 metre = 39·370 inches, we have approximately 1 metre = $39\frac{3}{8}$ inches. This may be remembered easily in two forms: 1 metre = 3 feet $3\frac{3}{8}$ inches, or 32 metres = 35 yards.

Hence

To reduce metres	to	yards	multiply by	$1 + \frac{1}{12} + \frac{1}{96}$.
„	„	ft.	„	$3 + \frac{1}{4} + \frac{1}{32}$.
„	cmm.	„ ins.	„	$\frac{4}{10} - \frac{1}{200} - \frac{1}{800}$.
„	yds.	„ metres	„	$1 - \frac{1}{10} + \frac{1}{70}$.
„	ft.	„ metres	„	$\frac{3}{10} + \frac{1}{200} - \frac{1}{5000}$.
„	ins.	„ cmm.	„	$\frac{10}{4} + \frac{4}{100}$.

EXAMPLE.—*Reduce* (i) 5·6342 *metres to yards:* (ii) 4·378 *centimetres to inches, and check each result by reduction to the original denomination.*

(i) 5·6342	6·1624	(ii) 4·378	1·724
·46952	·61624		
·05869		1·7512	4·310
	5·54616		68·96
6·1624	·08803	·0219	
		·0055	4·379
	5·6342		
		1·724*	

6. It is proposed to add from time to time in the *Gazette* to the above examples of approximation by successive simple fractions. It is hoped that readers who perceive possible improvements will kindly point them out. Suggestions as to other similar formulæ will be welcomed. It will be easy to give in some future number a concise list of those best worth preserving. A. LODGE.

[* In the examples marked with an asterisk the subtraction has been performed simultaneously with the addition by "complementary" or "shop" method (see *A. I. G. T. Report for* 1893, pp. 60, 63.)]

SOLUTIONS OF EXAMINATION QUESTIONS.

(The initials M., S., W., etc., are those of well-known public examinations—Matriculation, Sandhurst, Woolwich, etc.)

1. *Given the vertical angle, the altitude and the perimeter of a triangle, to construct it.* [S. Nov. 1893.]

Let AB = perimeter,
H = altitude,
CDE = vertical angle.

Produce CD to F and draw DG bisecting EDF.

On AB describe an arc APB containing an angle equal to CDG.

Draw a parallel to AB at a distance from it equal to H, cutting the arc in P. Join AP, PB.

At P draw PQ, PR, making angles APQ, BPR equal to PAB, PBA respectively, and meeting AB in QR. Then PQR is the required triangle.

$\because$ APB = CDG,
$\therefore$ PAB + PBA = GDE,
$\therefore$ APQ + BPR = GDE,
$\therefore$ QPR = CDE.

It is easily seen to have the required perimeter and altitude. This affords an instructive example of the method of analysis and synthesis to be adopted when a problem does not yield readily to the method of intersection of Loci. For some apposite remarks see Petersen's *Methods and Theories* (Sampson, Low & Co.)

The analysis was as follows :—

Drawing some triangle PQR to represent the one required, we produce QR both ways, making QA, RB equal to QP, RP respectively. We join P to A and B to see if we can infer any connection between AB, which is of given magnitude and the vertex P. Examination of the figure shows that APB consists of P, $\frac{1}{2}$Q, and $\frac{1}{2}$R, and therefore exceeds $\frac{1}{2}$P by a right angle. If then P be taken as the given vertical angle, APB is also fixed. Hence, if AB is kept fixed in magnitude, P lies on a certain circular arc. Hence the construction.

Another solution of some interest depends on the fact that the base of a triangle of constant perimeter, whose vertical angle is given in magnitude and position, envelops a fixed arc of a circle (one of the ex-circles of the triangle).

The *altitude* being given, the base must also touch another given circle.

Hence the following solution :—

Along the straight lines containing the given vertical angle cut off PH, PK, each equal to half the perimeter. Describe a circle HLK touching PH, PK at H and K. Draw a common tangent QLR to this circle and that described with centre P and radius equal to the altitude, cutting OH, OK in QR. Then PQR is the required triangle (Fig. ii).

2. *Factorise.*—$1 + y(1 + x)^2(1 + xy)$.

[M. Jan. 1894; S. June 1893.]

Having noticed that no power of y higher than the square occurs, we arrange the given expression in powers of y thus—

$$1 + (1 + x)^2 y + x(1 + x)^2 y^2.$$

To obtain factors of this quadratic expression in y we have only to guess two expressions, whose *product* is $x(1 + x)^2$ and whose *sum* is $(1 + x)^2$.

These are easily seen to be $1 + x$ and $x(1 + x)$. Hence the required factors are $1 + (1 + x)y$ and $1 + x(1 + x)y$.

From experience with pupils we believe many candidates found more difficulty than they expected in trying to find these factors. Students may sometimes obtain a hint for factors of a general expression like the above by examining some special case. Thus, if we put $y = 1$ in the above expression we get $1 + (1 + x)^3$, which factorises as easily as $(2 + x)(1 + x + x^2)$.

Writing this $(1 + 1 + x)(1 + x + x^2)$, since the absolute term must in each factor be unity we obtain a strong hint as to the general factors.

3. *Produce a given straight line* AB *to a point* C, *so that the rectangle* AB, AC *may be equal to the square on* BC.

[S. June 1893.]

This was attached to II. 11, from which it differs only in requiring C to be in AB produced instead of in AB itself. It is pointed out in the *A. I. G. T. Elements of Plane Geometry* (p. 133) that a slight modification of the construction in II. 11 yields a solution of the new problem. Thus the *internal* position of C having been found by the construction indicated in Fig. i., the external one is found by that in Fig. ii., FD instead of FB being produced to G, the remaining construction remaining unaltered, and the proof nearly so. The required construction may of course be found by the solution of a quadratic equation, but an examination of the figure may suggest another method of obtaining it. In the course of the demonstration of II. 11 we prove that DG is divided at B in the same manner as it is required to divide AB at C, which suggests (what may easily be verified) that OG = OA.

Hence, *if we are given* BG, we can find D in GB produced such that GB, GD = square on BD by drawing the squares BG, HC, and producing OC to A, so that OA = OG, and then describing the square BAED, whence the solution given, the letters A, B, C, K, H, G, D, E, F of Fig. ii. corresponding to the letters G, B, D, K, E, A, C, H, O of Fig. i.

There is an interesting note by Leslie on the property used in the above analysis. After pointing out that GD is made up from AB by adding together lines GB, BD equal to the greater segment (BC) and the whole line, he proceeds :—

"Hence the peculiar division of any line being once obtained, a series of other lines all possessing the same property may readily be found by repeated additions. Thus, let AB be so cut that the square of BC is equal to the rectangle BA, AC. Make respectively BD = BA, DE = DC, EF = EB, and FG = FD; the lines CD, BE, DF, and EG are divided at the points B, D, E, F such that in each of them the square of the larger part is equivalent to the rectangle contained by the whole and the smaller part. *Even if the section of* AB *were assumed at first inexact, the series of combinations would always approach to greater accuracy.*

. . . . . .

"Assuming the segments of the divided line as at first equal, and each denoted by 1, the following successive numbers will result from a continued summation—

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, etc.,

which are thus composed—

$1 + 2 = 3,\ 2 + 3 = 5,\ 3 + 5 = 8,\ 5 + 8 = 13,\ 8 + 13 = 21$, etc.

"These numbers form, therefore, the simplest recurring series, a kind of approximation which I find was first noticed in this actual case by Girard, an ingenious Flemish mathematician.

"Hence, if the original line contained 144 equal parts, its greater segment would include 89 and its smaller segment 55 of these parts, very nearly; but $55 \times 144 = 7920$, being only one less than 7921, the square of 89."

Girard's numbers are easily obtained by an algebraical investigation of II. 11.

Thus, representing the whole line by unity, and the greater segment by x, we have—

$$x^2 = 1 - x.$$
$$x(x + 1) = 1.$$
$$x = \frac{1}{1 + x}$$
$$= \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}}}},$$

whence the successive convergents are found to be—

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \text{ etc.}$$

[For a curious application of these numbers to a well-known geometrical paradox, see Rouse Ball's *Mathematical Recreations and Problems*, p. 35.]

4. O, O′ *are two points within a triangle* ABC *such that angles* OAB, OBC = *angles* O′AC, O′BA *respectively. Show that angles* OCA = *angles* O′CB.

[C. S. L. 1892; Pet. Schol. 1891.]

Draw OD, O′P perpendicular to BC; OE, O′Q to CA; OF, O′R to CA.

Join EF, QR, PQ, DE.

Because angles OFA, OEA are right,

∴ O, E, A, F are concyclic.

∴ OEF = OAB.

Similarly O′RQ = O′AC.

∴ OEF = O′RQ.

Similarly OFE = O′QR.

Hence triangles OEF, O′QR are equiangular.

∴ rectangle OE . O′Q = rectangle OF . O′R.

Similarly rectangle OF . O'R = rectangle OD . O'P.

∴ rectangle OE . O'Q = rectangle OD . O'P.

But angles DOE, PO'Q are equal, each being supplementary to BCA.

∴ triangles DOE, PO'Q are equiangular.

∴ ODE = O'QP.

Since quadrangles ODCE, O'PCQ are cyclic,

ODE = OCE and O'QP = O'CP.

∴ OCA = O'CB.

The above has been chosen as an example for solution on account of its connection with the "Recent Geometry of the Triangle."[1]

Straight lines like OA, O'A are said to be *isogonal conjugates* with respect to the angle BAC.

Points like O and O' are said to be *isogonal conjugate points* with respect to the triangle.

Note that the orthocentre and the circumcentre of a triangle are a pair of isogonal conjugate points.

Note further that the six points D, P, E, Q, R, F are concyclic.

For by similar triangles

AQ : O'A :: AF : OA,

and O'A : AR :: OA : AE.

∴ AQ : AR :: AF : AE.

∴ AE . AQ = AF . AR.

∴ E, Q, R, F are concyclic.

Also since the perpendicular bisectors of EQ, RF must each pass through the mid point H of OO', H must be the centre of the circle.

∴ HE = HQ = HR = HF.

Similarly HR = HF = HD = HP.

Hence D, P, E, Q, R, F are all on the same circle.

The circle is clearly the auxiliary circle of an ellipse inscribed in the triangle with O,O' for foci.

5. ABC *is any rectilineal angle less than the angle of an equilateral triangle, and* D *and* E *two points within it; find the points* F *and* G *in* AB, BC *such that* DF + FG + GE *is a minimum.* [W. Nov. 1893.]

We may often deduce the position of a maximum or minimum path by imagining one part of the path to remain fixed while the other part varies.

Thus in the present case if we keep EG fixed and make DF + FG a minimum, the line GF by a well-known theorem passes through the point H on the opposite side of AB to D and at an equal distance from it: passes in fact through *the image of* D *in* AB. Similarly if we kept DF fixed we could show that GF passes through *the image of* E *in* BC.

Hence the following construction:—

Draw DL, EM perpendicular to AB, BC and produce to H, K such that LH = DL and MK = EM. Join HK, cutting AB, BC in F and G, then DF + FG + GE is the required minimum path from D to E.

Take any other points P and Q in AB, BC.

[1] See a paper by Mr. R. F. Davis in the Fourteenth Report of the *A. I. G. T.*

Then by I. 4, HF = DF; HP = DP; KG = EG; KQ = EQ.

∴ DF + FG + GK = HK, and DP + PQ + QE = HP + PQ + QK.

But HP + PQ + QK > HK (two applications of I. 20).

∴ DF + FG + GE < DP + PQ + QE.

The condition as to the magnitude of the angle ABC ensures that HK cuts AB, BC. For by I. 4, angle HBL = angle DBL and angle KBM = angle EBM.

∴ HBL + ABC + KBM < three times ABC, and ∴ < two right angles.

For other examples of this method, see Milne's *Companion to Problem Papers.*

6. *If* BFC, CDA, AEB *be equilateral triangles described externally on the sides of a triangle* ABC *right angled at* A; *and if* AG *be drawn perpendicular to* BC; *show that triangles* BFG, CFG *are respectively equal to triangles* AEB, CDA. [W. Nov. 1893.]

The simplest proof is very obvious.

ΔBEA : ΔBFC :: BG : BC (VI. 31),

:: ΔBFG : ΔBFC (VI. 1).

But the theorem is an interesting one on account of the possibility of demonstrating it by the aid of Book I. only, as Viviani did (1668): "ignorans adhuc universalem propositionem trigesimam primam de similibus figuris ab Euclide in Elementorum libro VI. allatam."

For his solution see Eighteenth Report of *A. I. G. T.* 1892 (*Geometrical Odds and Ends*).

It is also given in Leslie's *Elements*, where it is ascribed to Torricelli.

QUESTIONS FOR SOLUTION BY STUDENTS.

(*As these are well within "scholarship" range, it is hoped that they may form a useful "problem paper."*)

1. ABC are three points on a circle whose centre is O such that arc AB is double of arc BC. If AH, BK are drawn perpendicular to OB, OC show that HK is parallel to AB.

2. PHQ, POQ are two triangles on opposite sides of the same base PQ such that angles QHP, HPQ = angles OQP, QPO respectively. Any parallel to OQ cuts PO, PQ in LM. Show that QL, HM intersect on a fixed circle.

3. OP, OQ are tangents to a circle; PO is produced to F so that QF = PQ. From a point M on PQ, MK, ML are drawn parallel to OP, OQ meeting QF, OP in K, L. Show that PK, QL intersect on the circle.

4. In an ellipse two radii CL, CM are drawn at right angles to each other, and CQ is perpendicular to LM. Show geometrically that the locus of Q is a circle.

5. Given $17^2 = 12^2 \cdot 2 + 1$, so that $\sqrt{2} = 17 \div 12$ approximately, deduce other closer approximations of the same nature without using continued fractions.

6. In a letter to the *English Mechanic* Lord Grimthorpe gives the following numbers for the construction of approxi-

mately regular pentagons: *If side* = 13 *then the diagonal* = 21 *and the radius of the circumcircle* = 11. In the *Illustrated Carpenter and Builder* I find that *the "mitre" for such a pentagon is* $13\frac{3}{4} \times 10$. Show that there exists a simple geometrical connection between the two results.

7. The general formula for three integers which may be taken to represent the lengths of the sides of a right-angled triangle is well known to be $(x^2+y^2)^2 = (x^2-y^2)^2 + (2xy)^2$. Show that such sets of lines occur simply in the geometry of the parabola.

8. A series of positive numbers $u_1, u_2, u_3, \ldots u_n$ is constructed under the condition $u_n = u_{n-1} + u_{n-2}$, u_1 and u_2 being taken arbitrarily. Find the limiting value of the fraction $u_n \div u_{n-1}$ as n increases indefinitely.

9. PN is the ordinate of an ellipse. Draw any straight line CH from C equal to the semi-major axis. Draw AK parallel to HN to meet CH in K. Produce CN to T, making CT = CK. Then PT is the tangent at P.

W. J. GREENSTREET, M.A.

10. Show that with the same constant and pole of inversion with which either of two given circles may be inverted into the other, their circle of similitude inverts into their radical axis. H. D. ELLIS, M.A.

11. In a triangle ABC it is given that $AB - AC = \frac{1}{2}BC$. If D be taken or BC such that $BD = \frac{1}{4}BC$ show that angle ACB is double of angle ADC.

12. A straight line cuts a conic in P, P′ and the directrix in Z. From any point K on it is drawn perpendicular, KI to the directrix, and a parallel KUU′ to ZS, meeting SP, SP′ in U, U′. Prove from the definition of a conic that $SU = SU' = e\text{KI}$.

Hence deduce Adam's property and show that a conic is in perspective with a circle whose centre is at the focus.

13. D is the mid point of the base BC of a triangle ABC. Any straight line CFGE through C meets AD in F, AB in G, and the parallel through A to BC in E. If CH be drawn perpendicular to AB, show that AB bisects the angle EHF.

14. Give a simple geometrical demonstration that if the sides of a triangle are to one another as 4 : 5 : 6, then the largest angle is double the smallest.

15. AB is a given straight line divided into two parts at the point C. Show how to find a point D in AC such that the square on AD shall be equal to the rectangle CD, BD.

16. Give a *simple geometrical* demonstration that

$$\sin 3A = 3 \sin A - 4 \sin^3 A,$$

and

$$\cos 3A = 4 \cos^3 A - 3 \cos A.$$

17. Solve *geometrically* the equation $\cos\theta = \tan\theta$. Show the connection between the above and the following Diophantine problem:—To find three numbers in continued proportion such that the square of the greatest shall be equal to the sum of the squares of the other two.

[*The Editor would be glad to receive contributions to this column.*]

L'INTERMÉDIAIRE DES MATHÉMATICIENS.

We have received the first number of a new journal, with the above title, which promises to be of great utility. To give an idea of its aim we cannot do better than quote from the preface: "Notre but essentiel est de fournir aux personnes qui cultivent habituellement les mathématiques, ou qui s'y intéressent, des renseignements sur des sujets se rapportant à leurs études, des solutions à des questions posées ou des indications bibliographiques . . . on ignore souvent ce qui se passe et ce qui se fait dans une branche voisine de celle dont on s'occupe particulièrement; aussi une question, de la solution de laquelle on aurait besoin, peut être très difficile pour celui qui désire cette solution, ou bien exigerait de sa part de longues recherches et une grande perte de temps, alors qu'une autre personne la considère, et avec raison à son point de vue, comme tout à fait simple.

"Mettre en rapport les deux personnes dont il s'agit c'est donc rendre service à la Science et contribuer à ses progrès en économisant des efforts inutiles."

We learn from the number before us that up to 5th December mathematicians had responded to the appeal of the editors, M. M. C.-A. Laisant and É. Lemoine, by sending in 217 questions and 50 answers. The first number contains 38 questions and 8 answers. To give an idea of the wide range of subjects to which the editors open their columns we quote the following questions:—

"On demande quelles sont les meilleures œuvres sur *l'enseignement primaire* de la Géométrie, les meilleurs Cours écrits dans les langues française, anglaise, espagnole, italienne et généralement adoptés."

"Démontrer que si q est dans le voisinage de 1 on a l'égalité asymptotique—

$$1 + 2q + 2q^4 + 2q^9 + = \sqrt{\pi/(1-q)}.$$"

"Trouver l'équation intrinsèque générale des cubiques planes et s'en servir pour la classification de ces courbes."

L'Intermédiaire is to appear once a month; the annual subscription is fixed, very moderately, at 5f.

We wish the editors every success.

BOOKS RECEIVED.

Modern Plane Geometry. By G. RICHARDSON, M.A., and A. RAMSEY, M.A. (Macmillan and Co.) The twofold object of this handy little treatise is, according to the authors, "to serve, in the first place, as a sequel to Euclid, or rather to the Treatise on Elementary Plane Geometry, issued by the *A. I. G. T.*; and secondly, as a systematic means of procedure from Euclidean Geometry to the higher descriptive Geometry of Conics and of imaginary points. . . . There has practically been no departure from the *Syllabus* of the *A. I. G. T.*"

Analytical Geometry for Beginners. Part I.—"The Straight Line and Circle." By T. G. VYVYAN, M.A. (G. Bell and Sons.)

Eccentric Circle

Fig. 1.

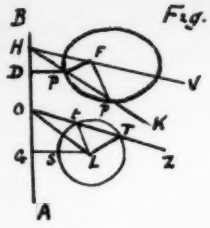
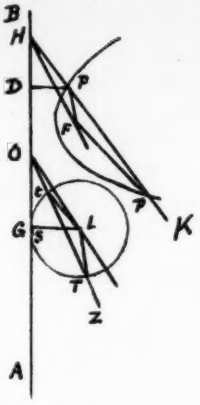


Fig 2.



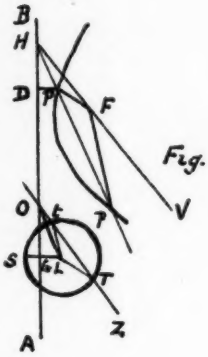
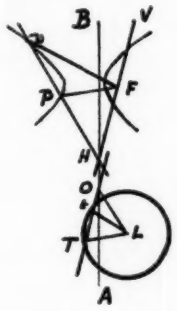


Fig. 3.

Fig 4



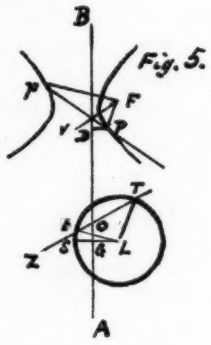
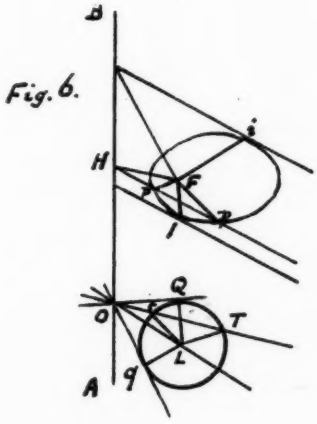


Fig. 5.



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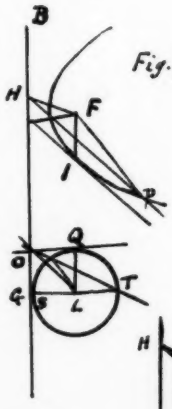

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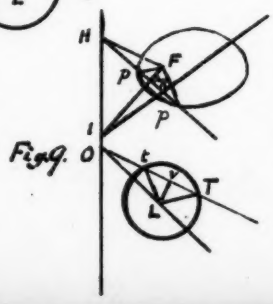


Fig. 9.

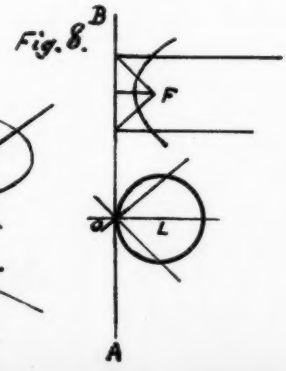
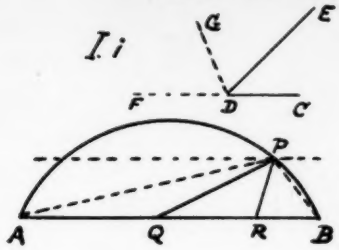


Fig. 8.

Solutions

I. i



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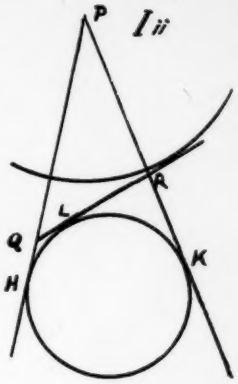

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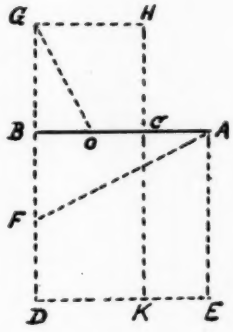

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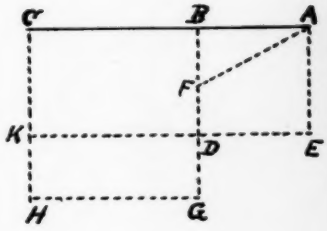

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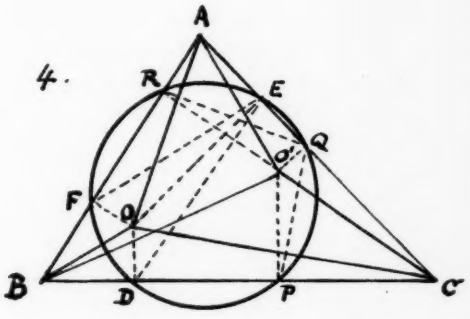
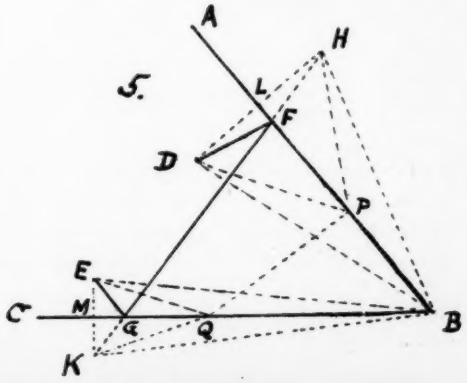
5.